

(1 + 1)-Dirac Particle with Position-Dependent Mass in Complexified Lorentz Scalar Interactions: Effectively \mathcal{PT} -Symmetric

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Abstract The effect of the built-in supersymmetric quantum mechanical language on the spectrum of the (1 + 1)-Dirac equation, with position-dependent mass (PDM) and complexified Lorentz scalar interactions, is re-emphasized. The signature of the “quasi-parity” on the Dirac particles’ spectra is also studied. A Dirac particle with PDM and complexified scalar interactions of the form $S(z) = S(x - ib)$ (an inversely linear plus linear, leading to a \mathcal{PT} -symmetric oscillator model), and $S(x) = S_r(x) + iS_i(x)$ (a \mathcal{PT} -symmetric Scarf II model) are considered. Moreover, a first-order intertwining differential operator and an η -weak-pseudo-Hermiticity generator are presented and a complexified \mathcal{PT} -symmetric periodic-type model is used as an illustrative example.

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1 Introduction

A fermion bound to move in the x -direction (i.e., $p_y = p_z = 0$) mandates the decomposition of the (3+1)-dimensional Dirac equation into two (1 + 1)-dimensional equations with two-component spinors and 2×2 Pauli matrices. Whilst the scalar $S(x)$, and vector $V(x)$, potentials preserve their Lorentz structures (i.e., the former is added to the mass term of Dirac equation while the minimal coupling is used, as usual, for the latter), the angular momentum and spin are absent in the process. Manifesting, in effect, a mathematically easily assessable and physically more transparent exploration of the (1 + 1)-Dirac world.

Nevertheless, the supersymmetric quantum mechanical terminology is realized (cf., e.g. [1–9]) as a *hidden/built-in symmetry* in the (1 + 1)-dimensional Dirac equation with “the mainly motivated by the MIT bag model of quarks” Lorentz scalar potential [10, 11]. For example, Nogami and Toyama [1] have reported that the associated supersymmetric

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Schrödinger Hamiltonians H_1 and H_2 share the same energy spectrum including the lowest states unless Dirac equation allows a zero-mode (i.e., zero-energy bound-state). Moreover, Jackiw and Rebbi [9] have reported that if the Lorentz scalar potential is localized (i.e., $S(x) \rightarrow 0$ for $x \rightarrow \pm\infty$) no zero-mode is allowed. That is, only for some Lorentz scalar potentials exhibiting certain *topological* trends, Dirac equation admits zero-mode.

Although the practical/experimental determination of the full spectrum is often proved impossible, exact solvability of quantum mechanical models (relativistic and non-relativistic) remains inviting and desirable. On the *exact-solvability* methodical side, however, attention was (by large) paid to the non-relativistic Schrödinger equation, whereas the relativistic Klein–Gordon and Dirac equations are left unfortunates. Not only within the recent revival of the unusual non-Hermitian complexified Hamiltonians' settings [12–40], but also within the usual Hermitian ones including those with position-dependent mass (PDM) [41–54].

In their pioneering generalization of the non-relativistic quantization recipe (i.e., \mathcal{PT} -symmetric Hamiltonians, where \mathcal{P} denotes parity and the complex conjugation \mathcal{T} mimics time reversal.), Bender and Boettcher [12–14] have suggested a tentative weakening/relaxation of Hermiticity as a necessary condition for the reality of the spectrum (i.e., the reality of the spectrum is secured by the exactness of \mathcal{PT} -symmetry). However, Mostafazadeh [24–27] has introduced a broader class of the so-called pseudo-Hermitian Hamiltonians with real spectra (within which \mathcal{PT} -symmetric Hamiltonians form a subclass). He has, basically, advocated the “user-friendly” consensus that neither Hermiticity nor \mathcal{PT} -symmetry serve as necessary conditions for the reality of the spectrum of a quantum Hamiltonian [12–40]. Yet, the existence of the real eigenvalues is realized to be associated with a non-Hermitian Hamiltonian provided that it is an η -pseudo-Hermitian, $\eta H = H^\dagger \eta$, with respect to the nontrivial “metric” intertwining operator η ($= O^\dagger O$, for some linear invertible operator $O : \mathcal{H} \rightarrow \mathcal{H}$, where \mathcal{H} is the Hilbert space). Furthermore, one may rather choose to be disloyal to the Hermiticity (cf., e.g., Bagchi and Quesne [34, 35]), and “linear” and/or “invertible” (cf., e.g., Solombrino [36] and Mustafa and Mazharimousavi [37, 38]) conditions on the intertwiner η , and hence relaxing H to be an η -weak-pseudo-Hermitian.

On the one (among others, some of which are readily mentioned above) of the main stimulants/inspirations of the present article, we may re-collect that quantum particles endowed with PDM constitute useful models for the study of many physical problems. In particular (but not limited to), they are used in the energy density many-body problem, in the determination of the electronic properties of semiconductors and quantum dots (cf., e.g., the sample of references in [37–54]), etc.

In the forthcoming text, we shall focus (in addition to the $(1 + 1)$ -Dirac particle with PDM in complexified Lorentz scalar interactions) on two main spectral phenomenological properties. Namely the energy-levels crossings (manifested by the “quasi-parity” settings of Znojil's [35] attractive/repulsive-like core) and the related effects to the hidden/built-in supersymmetric terminology in the $(1 + 1)$ -Dirac equation. Both in the usual Hermitian and the unusual complexified non-Hermitian settings.

The organization of this article is in order. In Sect. 2, we discuss the $(1 + 1)$ -Dirac equation with PDM and a Lorentz-scalar interaction and re-emphasize Nogami's and Toyama's [1] hidden/built-in supersymmetric language. We report, in Sect. 3, some consequences of a complexified non-Hermitian \mathcal{PT} -symmetric Lorentz scalar potentials belonging to two different classes: $S(x) \rightarrow S(x - ib) = S(z)$; $x, b \in \mathbb{R}$, $z \in \mathbb{C}$ and $S(x) = S_r(x) + iS_i(x)$; $S_r(x), S_i(x) \in \mathbb{R}$; an inversely linear plus linear and a Scarf II models, respectively. In Sect. 4, we explore one possibility of η -weak-pseudo-Hermiticity generators via a first-order intertwining differential operator. We exemplify this possibility by an η -weak-pseudo-Hermitian \mathcal{PT} -symmetric periodic-type model. We conclude in Sect. 5.

2 (1 + 1)-Dirac Equation with a Position Dependent Mass and a Lorentz Scalar Interactions

In the presence of a time-independent position-dependent mass $m(x)$, and a Lorentz scalar interaction $S(x)$, the (1 + 1)-dimensional time-independent Dirac equation (in $c = \hbar = 1$ units) reads

$$H_D \Psi(x) = E \Psi(x), \quad H_D = \alpha p + \beta[m(x) + S(x)], \tag{1}$$

where $p = -i\partial_x$, α and β are the usual 2×2 Pauli matrices satisfying the relations $\alpha^2 = \beta^2 = 1$ and $\{\alpha, \beta\} = 0$, E is the energy of the Dirac particle, and $\Psi(x)$ is the two-component spinor. Equation (1) with

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}, \quad \alpha = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

reduces (with $M(x) = m(x) + S(x)$) to

$$\begin{pmatrix} 0 & -\partial_x + M(x) \\ \partial_x + M(x) & 0 \end{pmatrix} \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} = E \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \tag{3}$$

which decouples, in turn, into

$$[-\partial_x + M(x)]\psi_-(x) = E\psi_+(x), \tag{4}$$

$$[\partial_x + M(x)]\psi_+(x) = E\psi_-(x). \tag{5}$$

This would, with $\omega = \pm 1$, imply a Schrödinger-like equation

$$\{-\partial_x^2 + V_\omega(x)\}\psi_\omega(x) = \lambda_\omega \psi_\omega(x), \tag{6}$$

$$V_\omega(x) = M(x)^2 - \omega M'(x), \quad \lambda_\omega = E_\omega^2, \tag{7}$$

where prime denotes derivative with respect to x (i.e., ∂_x). Nevertheless, a built-in supersymmetric quantum mechanical language is obvious in (7). That is, if the superpotential is defined as $W(x) = -M(x)$, then the supersymmetric partner potentials are given by

$$V_\omega(x) = M(x)^2 - \omega M'(x) = W^2(x) \pm W'(x).$$

In this case, one would label $H_- = -\partial_x^2 + V_-(x)$ and $H_+ = -\partial_x^2 + V_+(x)$ as the two partner Hamiltonians (cf., e.g., Alhaidari [51], and Sinha and Roy [3]). Of course, such supersymmetric language would leave its fingerprints/signature on the spectrum, as shall be witnessed in the forthcoming experiments with both Hermitian and non-Hermitian models.

3 Consequences of Complexified Non-Hermitian Lorentz Scalar Interactions

In this section, we consider two cases: a Dirac particle with $S(x) \rightarrow S(x - ib) = S(z)$ (where $\mathbb{C} \ni z = x - ib$, $\mathbb{R} \ni x \in (-\infty, \infty)$, and $\mathbb{R} \ni \text{Im } z = -b < 0$, i.e., a simple constant downward shift of the coordinate is considered), and Dirac particle with $S(x) = S_r(x) + iS_i(x)$, where $S_r(x), S_i(x) \in \mathbb{R}$.

3.1 Dirac Particle with a Complexified Lorentz Scalar $S(x) \rightarrow S(z) \equiv S(x - ib)$:
 a \mathcal{PT} -Symmetrized Znojil’s Oscillator

A Dirac particle under the influence of $M(z(x)) = m(x) + S(z(x))$ (where $z = x - ib$, $x \in (-\infty, \infty)$, and $\mathbb{R} \ni \text{Im } x = -b < 0$, i.e., a simple constant downward shift of the coordinate is considered) would result in recasting (6) and (7) as

$$\{-\partial_z^2 + V_\omega(z)\}\psi_\omega(z) = \lambda_\omega \psi_\omega(z), \quad V_\omega(z) = M(z)^2 - \omega M'(z). \tag{8}$$

Then, a Dirac particle endowed with a mass function of the form $m(x) = Bx/4$, $\mathbb{R} \ni B \geq 0$, under the influence of a complexified non-Hermitian Lorentz scalar interaction $S(z) = \frac{B}{4}z + \frac{A}{z} - i\frac{B}{4}b$ would imply

$$M(z) = \frac{B}{2}z + \frac{A}{z}. \tag{9}$$

Which, in effect, yields two complexified non-Hermitian \mathcal{PT} -symmetric partner potentials

$$V_\omega(z) = \frac{1}{4}B^2z^2 + \frac{A(A + \omega)}{z^2} + B\left(A - \frac{\omega}{2}\right). \tag{10}$$

Moreover, it should be noted that $V_+(z)$ represents a \mathcal{PT} -symmetric complexified oscillator perturbed by a “shifted by a constant” Znojil’s [35] repulsive/attractive core, i. e., with the parametric choice $A = \alpha - \frac{1}{2}$, $\alpha \geq 0$, one gets

$$\frac{A(A + 1)}{z^2} = \frac{(\alpha^2 - \frac{1}{4})}{z^2}, \tag{11}$$

cf., e.g. Mustafa and Znojil [15–23], and $V_-(z)$ represents a \mathcal{PT} -symmetric oscillator perturbed by a shifted/rescaled, say, Znojil’s [35] repulsive/attractive core, i.e.,

$$\frac{A(A - 1)}{z^2} + B\left(A - \frac{1}{2}\right) = \frac{(\alpha^2 - \frac{1}{4}) - 2(\alpha - \frac{1}{2})}{z^2} - \alpha B. \tag{12}$$

Under these settings, one would map Znojil’s results [35], taking into account our discussion on the supersymmetric-like partner potentials in (10), and obtain

$$\lambda_{+,q} = \frac{B}{2}(4n + 2q\alpha + 2\alpha), \quad n = 0, 1, 2, \dots, \tag{13}$$

$$\lambda_{-,q} = \frac{B}{2}[4n + (\alpha + 1)(2 - 2q)], \quad n = 0, 1, 2, \dots \tag{14}$$

We observe the supersymmetric language “signature” in $\lambda_{+,q=+1} = \lambda_{-,q=+1}$ for even quasi-parity and $\lambda_{+,q=-1} + const. = \lambda_{-,q=-1}$ for odd quasi-parity, i.e.,

$$\lambda_{+,q=+1} = \lambda_{-,q=+1} = 2B(n + \alpha), \tag{15}$$

and

$$\lambda_{-,q=-1} = \lambda_{+,q=-1} + 2B = 2B(n + 1). \tag{16}$$

Leading, in effect, (with $E_{+,q} = +\sqrt{\lambda_{+,q}}$ and $E_{-,q} = -\sqrt{\lambda_{-,q}}$) to

$$E_{+,q} = \begin{cases} E_{+,q=+1} = +\sqrt{2B(n + \alpha)}, \\ E_{+,q=-1} = +\sqrt{2Bn}, \end{cases} \tag{17}$$

$$E_{-,q} = \begin{cases} E_{-,q=+1} = -\sqrt{2B(n + \alpha)}, \\ E_{-,q=-1} = -\sqrt{2B(n + 1)}. \end{cases} \tag{18}$$

Yet, the *energy-levels crossing* phenomenon (a quasi-parity signature on the spectrum above) is observed unavoidable. That is, the two sets of energies in (17) cross with each other when

$$E_+(n = n_1, q = +1) = E_+(n = n_2, q = -1) \implies n_2 - n_1 = \alpha \tag{19}$$

and the sets of energies in (18) cross with each other when

$$E_-(n = n_3, q = +1) = E_-(n = n_4, q = -1) \implies n_4 - n_3 = \alpha - 1. \tag{20}$$

3.2 Dirac Particle with a Complexified Lorentz Scalar $S(x) = S_r(x) + iS_i(x)$: a \mathcal{PT} -symmetric Scarf II Model

In this section we consider a class of a complexified Lorentz scalar models of the form $S(x) = S_r(x) + iS_i(x)$ and position-dependent mass $m(x) \neq 0$ in $M(x) = m(x) + S(x)$. For simplicity of calculations, we take

$$M(x) = \tilde{M}(x) + iS_i(x), \quad \tilde{M}(x) = m(x) + S_r(x). \tag{21}$$

If we assume that $\tilde{M}(x) = (A + B) \tanh x$ and $S_i(x) = -(A - B) \operatorname{sech} x$ then

$$M(x) = (A + B) \tanh x - i(A - B) \operatorname{sech} x \tag{22}$$

and consequently the corresponding supersymmetric \mathcal{PT} -symmetric partner potentials are given by

$$V_\omega(x) = -C_1 \operatorname{sech}^2 x - iC_2 \operatorname{sech} x \tanh x + (A + B)^2 \tag{23}$$

where $C_1 = 2(A^2 + B^2) + \omega(A + B)$ and $C_2 = (2A + 2B + \omega)(A - B)$. It is obvious that $V_\omega(x)$ is the well known complexified \mathcal{PT} -symmetric Scarf II model. Moreover, it should be noted that $V_+(x)$ and $V_-(x)$ imitate the pseudo-supersymmetric \mathcal{PT} -symmetric partner potentials $U_2(x)$ and $U_1(x)$, respectively, reported in (38–40) by Sinha and Roy [3]. The solution of which can be easily mapped into the above model, by taking the constant mass in Sinha and Roy [3] equals zero, to obtain

$$E_{+,n}(A, B) = +\sqrt{2(A + B)(n + 1) - (n + 1)^2}, \quad n = 0, 1, 2, \dots \tag{24}$$

$$E_{-,n}(A, B) = -\sqrt{2(A + B)n - n^2}, \quad n = 0, 1, 2, \dots \tag{25}$$

However, it is obvious that

$$E_{+,n}(A, B) \in \mathbb{R} \iff [2(A + B) - 1] \geq n \tag{26}$$

and

$$E_{-,n}(A, B) \in \mathbb{R} \iff 2(A + B) \geq n. \tag{27}$$

This result in effect documents the fact that \mathcal{PT} -symmetry is not an enough condition to guarantee the reality of Dirac spectrum but rather it should be complemented by the condition $E_n^2 \geq 0$. Moreover, *energy-levels crossing* phenomenon introduces itself (in this case, of course, not as a quasi-parity effect but rather as a spectral property) in the following scenario: the energy levels in the set (24) perform *energy-levels crossing* among each other when

$$E_{+,n_1}(A, B) = E_{+,n_2}(A, B) \implies n_1 + n_2 = 2(A + B - 1), \tag{28}$$

and similar trend is also obvious in (25) when

$$E_{-,n_3}(A, B) = E_{-,n_4}(A, B) \implies n_3 + n_4 = 2(A + B). \tag{29}$$

4 Consequences of η -Weak-Pseudo-Hermiticity via a First-Order Intertwiner

A complexified non-Hermitian Lorentz scalar interaction, $S(x) = S_r(x) + iS_i(x)$, where $S_r(x), S_i(x) \in \mathbb{R}$, would result in

$$\text{Re } V_\omega(x) = m(x)^2 + S_r(x)^2 - S_i(x)^2 + 2m(x)S_r(x) - \omega[m'(x) + S'_r(x)], \tag{30}$$

$$\text{Im } V_\omega(x) = 2m(x)S_i(x) + 2S_i(x)S_r(x) - \omega S'_i(x). \tag{31}$$

We may now work with a Schrödinger-like non-Hermitian Hamiltonian operator $\tilde{H}_\omega = -\partial_x^2 + V_\omega(x)$ with the eigenvalues $\lambda_\omega = E^2$. Then \tilde{H}_ω is an η -weak-pseudo-Hermitian (admitting real eigenvalues $\lambda_\omega = E^2 \in \mathbb{R}$) with respect to the first-order Hermitian intertwiner

$$\eta = -i \partial_x + G(x), \tag{32}$$

where $G(x) \in \mathbb{R}$, if it satisfies the intertwining relation $\eta \tilde{H}_\omega = \tilde{H}_\omega^\dagger \eta$ (it is not difficult to show that $(\eta \tilde{H}_\omega)$ is Hermitian too).

Under such η -weak-pseudo-Hermiticity settings, the intertwining relation would result in

$$\text{Im } V_\omega(x) = -G'(x) \quad \text{and} \quad \text{Re } V_\omega(x) = -G(x)^2 \tag{33}$$

to yield, respectively,

$$-G'(x) = 2m(x)S_i(x) + 2S_i(x)S_r(x) - \omega S'_i(x), \tag{34}$$

$$-G(x)^2 = m(x)^2 + S_r(x)^2 - S_i(x)^2 + 2m(x)S_r(x) - \omega[m'(x) + S'_r(x)]. \tag{35}$$

Consequently, (34) implies

$$m(x) + S_r(x) = \frac{-G'(x) + \omega S'_i(x)}{2S_i(x)}. \tag{36}$$

Substituting (36) in (35) would yield

$$\left[\frac{-G'(x) + \omega S'_i(x)}{2S_i(x)} \right]^2 - \omega \left[\frac{-G'(x) + \omega S'_i(x)}{2S_i(x)} \right]' = [\omega S_i(x)]^2 - G(x)^2. \tag{37}$$

The simplest solution of which is given by (with $\omega = \pm 1$) the choice

$$G_\omega(x) = \omega S_i(x) \Rightarrow S_{i,\omega}(x) = \omega G(x) \Rightarrow S_r(x) = -m(x). \tag{38}$$

In the forthcoming experiment, we shall be interested in the family of complexified Lorentz scalar interactions of the form $S(x) = -m(x) + i S_i(x)$. With such settings in point, the Dirac Hamiltonian in (1) collapses into

$$H_D = \sigma_2 p + i \sigma_1 S_i(x) = \begin{pmatrix} 0 & -\partial_x + i S_i(x) \\ \partial_x + i S_i(x) & 0 \end{pmatrix}. \tag{39}$$

Consequently and without any loss of generality, one may very well recast our η -weak-pseudo-Hermitian Schrödinger-like Hamiltonian as

$$\tilde{H}_\omega = -\partial_x^2 + V_\omega(x) = -\partial_x^2 - G(x)^2 - i\omega G'(x). \tag{40}$$

4.1 An η -Weak-Pseudo-Hermitian \mathcal{PT} -Symmetric Periodic-Type Model

An η -weak-pseudo-Hermiticity generator of a periodic nature of the form

$$G(x) = -\frac{4}{3 \cos^2 x - 4} - \frac{5}{4} \tag{41}$$

would imply \mathcal{PT} -symmetric periodic-type effective potentials

$$V_\omega(x) = -G(x)^2 - i\omega G(x)' = \frac{1 - 30 \cos^2 x + 24}{9 (\cos^2 x - \frac{4}{3})^2} + i \frac{4\omega \sin 2x}{3(\cos^2 x - \frac{4}{3})^2} - \frac{25}{16} \tag{42}$$

which, in a straightforward manner, can be rewritten as

$$V_\omega(x) = -\frac{6}{(\cos x + 2i\omega \sin x)^2} - \frac{25}{16}. \tag{43}$$

It should be noted here that $V_+(x)$ is the \mathcal{PT} -symmetric periodic-type effective potential representing a “*shifted by a constant*” Samsonov–Roy’s [55] periodic potential model satisfying $V_\omega(x) = V_{-\omega}(-x)$. Hence, if we defined

$$\tilde{V}_\omega(x) = -\frac{6}{(\cos x + 2i\omega \sin x)^2}, \tag{44}$$

then (with \mathcal{P} denoting parity)

$$\mathcal{P} \tilde{V}_\omega(x) = \tilde{V}_\omega(-x) = \tilde{V}_{-\omega}(x).$$

Consequently, $\tilde{V}_\pm(x)$ and $\tilde{V}_\mp(x)$ mirror reflect each other. A result that provides a safe passage through the transformation $x \rightarrow y = -x$ and mandates

$$\tilde{H}_\pm = -\partial_x^2 + V_\omega(x) = -\partial_y^2 + V_{-\omega}(y).$$

The solution of which is reported for the interval $x \in (-\pi, \pi)$ (equivalently, $y \in (-\pi, \pi)$) with the boundary conditions $\psi_{n,\omega}(-\pi) = \psi_{n,\omega}(\pi) = 0$ as

$$\psi_{n,\omega}(x) = \left\{ [(16 - n^2) \cos x - \omega 2i(n^2 - 4) \sin x] \sin \left[\frac{n}{2}(\pi + \omega x) \right] \right\}$$

$$-6\omega n \sin x \cos \left[\frac{n}{2}(\pi + \omega x) \right] \} (\cos x + 2i\omega \sin x)^{-1} \quad (45)$$

and

$$E_{n,\omega} = \omega \sqrt{\lambda_\omega} = \omega \sqrt{\frac{n^2}{4} - \frac{25}{16}}, \quad n = 3, 4, 5, \dots \quad (46)$$

It should be reported here that the values of $n < 3$ are sacrificed for the sake of the reality of the spectrum.

5 Conclusion

In this work, the effect of the built-in supersymmetric quantum mechanical language on the structure of the decomposed $(1 + 1)$ -Dirac equation, with PDM and complexified Lorentz scalar interactions, is re-emphasized. In the process, the signature of the “quasi-parity” (manifested by Znojil’s attractive/repulsive-like core [35]) is also studied. In so doing, a “quasi-free” Dirac particle with PDM (an inversely linear plus linear), a Dirac particle with PDM and complexified scalar interactions, $S(z) = S(x - ib)$, $x, b \in \mathbb{R}$, $z \in \mathbb{C}$ (an inversely linear plus linear, leading to a \mathcal{PT} -symmetric oscillator model), and $S(x) = S_r(x) + iS_i(x)$, $S_r(x), S_i(x) \in \mathbb{R}$ (a \mathcal{PT} -symmetric Scarf II model) are considered. Moreover, a first-order intertwining differential operator and an η -weak-pseudo-Hermiticity generator are presented (a complexified \mathcal{PT} -symmetric periodic-type model is used).

In the light of our experience above we have observed that the associated supersymmetric signature on the spectrum of the $(1 + 1)$ -Dirac particle results in exact-isospectral (i.e., including the lowest states) partner Hamiltonians H_1 and H_2 for “even” quasi-parity, however, they share the same energy spectrum with a “missing” lowest state for “odd” quasi-parity. Nevertheless, we may report that the *energy-levels crossing* is only feasible among positive-energy states (i.e., above $E = 0$) or among negative-energy states (i.e., below $E = 0$), at least as long as our illustrative examples are concerned. We may also add that neither the exactness of \mathcal{PT} -symmetry nor pseudo-Hermiticity are enough conditions for the reality of the Dirac spectrum, they should be rather complemented by the condition $\mathbb{R} \ni E^2 > 0$. Finally, one may need to sacrifice some energy states for the sake of the reality of the Dirac particle spectrum.

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